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An existence result of a quasi-variational inequality associated to an equilibrium problem

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Abstract In this paper we consider a Walrasian pure exchange economy with utility function which is a particular case of a general economic equilibrium problem, without production. We assume that each agent is endowed with at least of a commodity, his preferences are expressed by an utility function and it prevails a competitive behaviour: each agent regards the prices payed and received as independent of his own choices. The Walrasian equilibrium can be characterized as a solution to a quasi-variational inequality. By using this variational approach, our goal is to prove an existence result of equilibrium solutions.

Keywords Competitive equilibrium · Variational and quasi-variational inequality · Mosco's convergence

1 Introduction

The full recognition of the general equilibrium concept can be attributed unmistakably to Leon Walras [24]. Taking into account more aspects of a real economy, he obtains a system of equations, which he calls the "equations of exchange": a solution to this system is an equilibrium for an exchange economy. The first rigorous result on the existence of general equilibrium is due to a series of papers by Wald [23]. Wald's papers were of forbidding mathematical depth, not only in the use of sophisticated tools, but also in the complexity of the argument. A help, finally, came from the development of a related line of research, as the Von Neumann's theory of games. He deduced [20], an existence theorem from a generalization of the Brouwer's fixed point theorem. With these foundations, plus the influence of the rapid

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development of linear programming on both the mathematical and the economic sides, it was perceived independently by a number of authors that existence theorems of greater simplicity and generality than Wald's ones were now possible. Some of these are McKenzie [15], Arrow and Debreu [1], Gale [10] and Nikaido [21].

Arrow and Debreu, in Ref. [1], by applying the fixed point theory, give an existence result when the data are convex, requiring that each agent starts out with a tradable quantity of every possible goods (that is the survivability assumption).
An alternative approach for the study of general economic equilibrium is performed by the variational inequality formulation. There are several papers that have been devoted to the study of this equilibrium by using the variational theory (e.g. see Ref. [3,11,12,18]); in particular, in Ref. [16] we can find a lot of references about the state of art of this topic. This theory arose in the seventies of the last century as an innovative and effective method to solve several equilibrium problems originated from mathematical physics as the Signorini's problem, the obstacle problem, the elastic-plastic torsion problem. It is still an open problem to decide who must be considered the founder between Fichera [9] and Stampacchia [22], who first dealt with variational inequality (e.g. see Ref. [14] for a survey).

In Ref. [3] Border elaborates a variational inequality formulation of a particular Walrasian price equilibrium problem, without any utility function. For this model, Nagurney in the book [18] (see e.g. also its complete bibliography), Nagurney and Zhao in Ref. [19] and Dafermos and Zhao in Ref. [6] use the variational approach for the study, analysis and computation of the equilibrium. In [11,12], Jofre, Rockafellar and Wets show how, introducing the Lagrange multipliers, the general economic equilibrium can be represented by a variational inequality problem. They, by means of truncation arguments, are able to establish the existence of a "virtual equilibrium", approximated by a classical Walrasian equilibrium. In order to achieve this existence result they assume the "strong survivability": every agent have, from the beginning, a positive quantity of every goods (see e.g. (A6) p. 13 [12]).

In this paper, we consider a competitive economic equilibrium problem and we study a pure exchange economy with l different goods and n agents. We suppose that the following survivability assumption holds: each agent is endowed with a positive quantity of at least one commodity [see e.g. Sect. 2 assumption (U5)]. In this market the agents' preferences are expressed by an utility function and it prevails a competitive behaviour, that is each agent regards the prices payed and received as independent of his own choices. Mathematically, we formulate the notion of this economic equilibrium in terms of excess demand function and of maximization of utility function related to each agent. Thanks to the given assumptions on the utility function we are able to guarantee that this economy is regulated by Walras' law. This equilibrium problem can be reformulated in terms of a quasi-variational inequality associated to the excess demand function and to the gradient of the utility function (for a bibliography of quasi-variational theory, see e.g. [4,8]). Our goal is to prove, by means of the set convergence in the Mosco's sense, an existence result related to the solutions of our quasi-variational inequality, that not satisfies the standard assumptions of usual existence theorems.

2 Walrasian pure exchange model

We consider a marketplace consisting of l different goods indexed by $j = 1, \dots, l$ and n agents indexed by $a = 1, \dots, n$. Each agent $a = 1, 2, \dots, n$ has an initial endowment vector:

$$e_a = (e_a^1, e_a^2, \dots, e_a^l) \in \mathrm{R}_+^l.$$

We denote by x_a^j the consumption by agent a of good j and represent with:

$$x_a = (x_a^1, x_a^2, \ldots, x_a^l) \in \mathrm{R}_+^l$$

the consumption choice vector and with:

$$x \equiv (x_1, x_2, \ldots, x_n)^T \in \mathrm{R}_+^{nl}$$

the consumption of market. In this economy there is only pure exchange, without production, that is the only activity that the agents can perform is to consume and/or trade their commodities with each other agent. We presume that the "law of one price" is fulfilled, that is, traders examine opportunities to the extent that each goods is sold and purchased at only one price. Each goods j, $j = 1, 2, \ldots, l$ associates with it a real positive number p^j representing its price and we denote by

$$p = (p^1, p^2, \ldots, p^l) \in \mathrm{R}_+^l$$

the price vector. We also presume a competitive behaviour, that is, agents do not perceive that they can have any influence over these market prices. Competitive equilibrium price vector, which we denote by $\overline{p}$, is the price at which every agent can simultaneously satisfy his desire to trade. As standard in economic theory, the choice by the consumer from a given set of alternative consumption vectors is supposed to be made in accordance with a preference scale for which there is an utility function:

$$u_a : \mathrm{R}_+^l \to \mathrm{R}$$

$$\mathrm{R}_+^l \ni x_a \to u_a(x_a) \in \mathrm{R}.$$

In this market, the objective of each of the agents is to maximize their utility by performing pure exchanges of the given goods. There are natural constraints that the consumers must satisfy: the wealth of a consumer is his initial endowment, and the total amount of goods that a consumer can acquire or buy is at most equal to his initial wealth, i.e. the goods that the consumer sells off. This means that, for all $a = 1, \ldots, n$ and for all $p \in P$:

$$u_a(\overline{x}_a) = \max_{x_a \in M_a(p)} u_a(x_a), \tag{1}$$

where

$$M_a(p) = \left\{ x_a \in \mathrm{R}^l : \ x_a^j \geq 0 \ \forall j = 1, \ldots, l, \ \sum_{j=1}^{l} p^j (x_a^j - e_a^j) \leq 0 \right\}, \quad \forall a = 1, \ldots, n,$$

and

$$p \in P = \left\{ p \in \mathrm{R}_+^l : \sum_{j=1}^{l} p^j = 1 \right\}.$$

For each $a = 1, .., n$ and $p \in P$, $M_a(p)$ is a closed and convex set of R_+^l.

We define a particular aggregate excess demand function:

$$z^j : \mathrm{R}_+^{nl} \to \mathrm{R}, \quad j = 1, 2, \ldots, l$$

$$x \to z^j(x) = \sum_{a=1}^{n} (x_a^j - e_a^j),$$

where $x_a^j - e_a^j$ is the individual excess demand by the agent a for the goods j. Grouping this components in the vector we introduce:

$$z(x) = (z^1(x), z^2(x), \dots, z^l(x)) \in \mathrm{R}^l.$$

Furthermore, for all $a = 1, \dots, n$, we assume that:

(U_1) u_a is strictly concave,

(U_2) $u_a \in C^1(\mathrm{R}^l_+)$ in the usual sense,

(U_3) $\forall x_a \in M_a(p) : \nabla u_a(x_a) \neq 0, \quad \forall p \in P$ and $\forall x_a \in \partial M_a(p) : \dfrac{\partial u_a(x_a)}{\partial x_a^s} > 0$, when $x_a^s = 0, \quad \forall p \in P$,

(U_4) $\lim\limits_{\substack{\|x_a\| \to +\infty, \\ x_a \in M_a(p)}} u_a(x_a) = -\infty$,

(U_5) Each agent is endowed with a positive quantity of at least one commodity

$$\forall\, a = 1, \dots, n \quad \exists j : \;\; e_a^j > 0,$$

and for every goods j there exists at least an agent a such that $e_a^j > 0$.

In our assumptions, for all $a = 1, \dots, n$, the maximization problem (1) has a unique solution for each $p \in P$, then it arises a function $\overline{x}_a(p)$ from P to R^l_+. So, we can define $z(x(p)) : P \to \mathrm{R}$ and in the following we will continue to denote with $z(p)$ the composite function $z(p) = z(x(p))$.

Then the competitive equilibrium condition of a pure exchange economic market takes the following form:

Definition 1 Let $\overline{p} \in P$ and $\overline{x}(\overline{p}) \in M(\overline{p}) = \Pi_{a=1}^n M_a(\overline{p})$. The pair $(\overline{p}, \overline{x}(\overline{p})) \in P \times M(\overline{p})$ is a competitive equilibrium if and only if:

for all $a = 1, \dots, n$

$$u_a(\overline{x}_a(\overline{p})) = \max_{x_a \in M_a(\overline{p})} u_a(x_a), \tag{2}$$

and for all $j = 1, 2, \dots, l$:

$$z^j(\overline{x}(\overline{p})) = \sum_{a=1}^n (\overline{x}_a^j(\overline{p}) - e_a^j) \leq 0. \tag{3}$$

The vector $\overline{p}$ is the competitive equilibrium price.

For sake of brevity in the sequel we will write $\overline{x}$ instead of $\overline{x}(\overline{p})$.

In the work [7] we have proved that, in our assumptions, the market is regulated by Walras' law:

$$\sum_{j=1}^l p^j(\overline{x}_a^j(p) - e_a^j) = 0 \qquad \forall p \in P, \;\; \forall a = 1, \dots, n, \tag{4}$$

hence it is possible reformulate the equilibrium in the following way:

A competitive equilibrium of a pure exchange economic market with utility function consists of a competitive equilibrium price vector $\overline{p} \in P$ *and a consumption vector* $\overline{x} \in \mathrm{R}^{nl}_+$ *such that:*

(a) *for all* $a = 1, \ldots, n, \overline{x}_a$ *is a solution to maximization problem* (2) *and*

$$\sum_{j=1}^{l} \overline{p}^j (\overline{x}_a^j - e_a^j) = 0. \tag{5}$$

(b) *For all* $j = 1, \ldots, l$*:*

$$\sum_{a=1}^{n} (\overline{x}_a^j - e_a^j) \begin{cases} \leq 0 & if \quad \overline{p}^j = 0 \\ = 0 & if \quad \overline{p}^j > 0. \end{cases} \tag{6}$$

Problem (2) states that the consumption choice vector x_a of the agent a must be such that his utility $u_a(x_a)$ is maximized, and the choice is subjected to the constraint that the amount that the agent a pays for acquiring the goods x_a, $\sum_{j=1}^{l} \overline{p}^j x_a^j$, is at most the amount that the agent receives for his initial endowment, $\sum_{j=1}^{l} \overline{p}^j e_a^j$. Condition (5) states that the amount that the agent a pays for acquiring the goods that maximized his utility: $\sum_{j=1}^{l} \overline{p}^j \overline{x}_a^j$, is equal to the amount that the agent received for his initial endowment: $\sum_{j=1}^{l} \overline{p}^j e_a^j$. Condition (6) states that the market is usually considered to be in equilibrium when, for a commodity, the supply equals the demand; but, there exists the possibility that at a zero price, the supply will exceed the demand. This is the classical case of the *free goods*.

In the work [7] we have proved that the competitive equilibrium for a pure exchange economic market is characterized as a solution to the quasi-variational inequality:

"*Find* $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *such that:*

$$\left\langle \sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \right\rangle_l + \sum_{a=1}^{n} \langle \nabla u_a(\overline{x}_a), x_a - \overline{x}_a \rangle_l \leq 0 \qquad \forall\, (p, x) \in P \times M(\overline{p})\text{"}, \tag{7}$$

in fact the following result holds:

Theorem 1 *The pair* $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *is a competitive equilibrium of a pure exchange economic market with utility function if and only if is a solution to quasi-variational inequality* (7).

Proof See e.g. [7]. □

3 Existence theorem

In this section we are concerned with the problem of the existence of the solutions to quasi-variational inequality (7).

We prove this result assuming that the operators $-\nabla u_a(x_a)$ are strongly monotone for all $a = 1, \ldots, n$ and for all $p \in P$:

$$\langle -\nabla u_a(x_a) + \nabla u_a(y_a), x_a - y_a \rangle \geq \nu ||x_a - y_a||^2 \qquad \forall x_a,\ y_a \in M_a(p). \tag{8}$$

We observe that the quasi-variational inequality (7) can be studied in the following way. First, we consider for all $p \in P$ and for all $a \in 1, \ldots, n$ the unique solution $\overline{x}_a(p)$ to the variational inequality:

"*Find* $\overline{x}_a(p) \in M_a(p)$ *such that:*

$$\langle -\nabla u_a(\overline{x}_a(p)), x_a - \overline{x}_a(p) \rangle_l \geq 0, \quad \forall x_a \in M_a(p);\text{"} \tag{9}$$

Then we solve the variational inequality:

"Find $\overline{p} \in P$ such that:

$$\left\langle -\sum_{a=1}^{n} (\overline{x}_a - e_a), p - \overline{p} \right\rangle_l \geq 0, \quad \forall p \in P." \tag{10}$$

The pair $(\overline{p}, \overline{x})$ clearly solve the quasi-variational inequality (7). We observe that from (8), because the operator results strongly monotone, the variational inequality (9) admits a unique solution. For the variational inequality (10), being P closed, convex and bounded, we have:

Theorem 2 ([13]) *If $\overline{x}_a(p)$ is a continuous function, then the variational inequality problem* (10) *admits a solution $\overline{p} \in P$.*

Then our goal is to show that $\overline{x}_a(p)$ is continuous on P. In order to achieve the continuity result we need to recall the concept of set convergence in the sense of Mosco (see also e.g. [2]).

Definition 2 ([17]) Let $(V, \|\cdot\|)$ be an Hilbert space $\mathbf{K} \subset V$ a closed, nonempty, convex set. A sequence of nonempty, closed, convex sets $\mathbf{K}_n$ converges to $\mathbf{K}$ as $n \to +\infty$, i.e. $\mathbf{K}_n \to \mathbf{K}$, if and only if

(M1) for any $H \in \mathbf{K}$, there exists a sequence $\{H_n\}_{n\in\mathrm{N}}$ strongly converging to H in V such that H_n lies in $\mathbf{K}_n$ for all n,

(M2) for any $\{H_\mathrm{n}\}_{n\in\mathrm{N}}$ weakly converging to H in V, such that H_n lies in $\mathbf{K}_\mathrm{n}$ for all n, then the weak limit H belongs to $\mathbf{K}$.

In our assumption we have the following result:

Lemma 1 *Let $p \in P$ be. Then, for all sequence $\{p_n\}_{n\in\mathrm{N}} \subset P$ converging to p, the sequence of sets $M_a(p_n)$ converges to $M_a(p)$ in Mosco's sense.*

Proof For readers' convenience we report the proof of the Mosco's convergence of the sequence $\{M_a(p_n)\}$ to $M_a(p)$, which first we proved in Ref. [7].

Let $p \in P$ fix and let $\{p_n\}_{n\in\mathrm{N}} \subset P$ be a sequence, such that $p_n \to p \in P$. First we prove that $M_a(p_n) \to M_a(p)$ in Mosco's sense, i.e. it is enough to show that (M1) and (M2) hold. For the first one, let $x_a(p) \in M_a(p)$ be fixed and let us pose:

$$I = \{j : x_a^j(p) > 0\} \subseteq \{1, 2, \ldots, l\}.$$

We consider the following sequence:

$$x_a(p_n) = x_a(p) - \eta_n \quad \forall n \in \mathrm{N}, \tag{11}$$

in other words:

$$x_a^j(p_n) = x_a^j(p) - \eta_n^j \quad \forall j = 1, 2, \ldots, l, \quad \forall n \in \mathrm{N},$$

where the sequence $\{\eta_n^j\}$ is such that:

$$\begin{cases} \eta_n^j = 0, & j \notin I \\ \eta_n^j = \eta_n, & j \in I, \end{cases}$$

with η_n converging to zero and, if $\sum\limits_{j\in I} p_n^j > 0$, satisfying

$$\frac{\sum_{j=1}^{l}(p_n^j - p^j)(x_a^j(p) - e_a^j)}{\sum_{j\in I} p_n^j} < \eta_n < \min_{j\in I}\{x_a^j(p)\}.$$

Let us verify that $x_a(p_n) \in M_a(p_n)\ \forall n \in \mathrm{N}$.

(a) If $\sum\limits_{j\in I} p_n^j = 0$, we observe that, because $p_n^j \geq 0$, we have $p_n^j = 0$ for all $j \in I$. Then, it results:

$$\begin{aligned}\sum_{j=1}^{l} p_n^j(x_a^j(p_n) - e_a^j) &= \sum_{j\in I} p_n^j(x_a^j(p_n) - e_a^j) + \sum_{j\notin I} p_n^j(x_a^j(p_n) - e_a^j)\\ &= \sum_{j\notin I} p_n^j(-e_a^j) \leq 0.\end{aligned}$$

(b) If $\sum\limits_{j\in I} p_n^j > 0$, it results:

$$\begin{aligned}\sum_{j=1}^{l} p_n^j(x_a^j(p_n) - e_a^j) &= \sum_{j=1}^{l} p_n^j(x_a^j(p) - \eta_n^j - e_a^j)\\ &= \sum_{j=1}^{l} p_n^j(x_a^j(p) - e_a^j) - \sum_{j=1}^{l} p_n^j\eta_n^j\\ &= \sum_{j=1}^{l} p^j(x_a^j(p) - e_a^j) + \sum_{j=1}^{l}(p_n^j - p^j)(x_a^j(p) - e_a^j)\\ &\quad - \sum_{j\in I} p_n^j\eta_n^j - \sum_{j\notin I} p_n^j\eta_n^j\\ &= \sum_{j=1}^{l} p^j(x_a^j(p) - e_a^j) + \sum_{j=1}^{l}(p_n^j - p^j)(x_a^j(p) - e_a^j)\\ &\quad -\eta_n \sum_{j\in I} p_n^j. \end{aligned} \tag{12}$$

Owing to $x_a(p) \in M_a(p)$:

$$\sum_{j=1}^{l} p^j(x_a^j(p) - e_a^j) \leq 0,$$

and by choosing of η_n:

$$\sum_{j=1}^{l}(p_n^j - p^j)(x_a^j(p) - e_a^j) < \eta_n \sum_{j\in I} p_n^j.$$

Then, from (12), we obtain:

$$\sum_{j=1}^{l} p_n^j(x_a^j(p_n) - e_a^j) < \eta_n \sum_{j\in I} p_n^j - \eta_n \sum_{j\in I} p_n^j = 0,$$

being $x_a(p_n) \geq 0 \ \ \forall n \in \mathrm{N}$.

Then, $x_a(p_n) \in M_a(p_n)\, \forall n \in \mathrm{N}$. Moreover, we have

$$\lim_{n\to+\infty} x_a(p_n) = \lim_{n\to+\infty} x_a(p) - \eta_n = x_a(p).$$

Hence the proof of the first condition (M1) is just obtained. For the second one, let $\{x_a(p_n)\}_{n\in\mathrm{N}}$ a fixed sequence, with $x_a(p_n) \in M_a(p_n) \ \ \forall n \in \mathrm{N}$. such that $x_a(p_n) \to x_a(p)$. Because

$$\sum_{j=1}^{l} p_n^j(x_a^j(p_n) - e_a^j) \leq 0, \quad x_a(p_n) \geq 0 \ \ \forall n \in \mathrm{N}$$

we get

$$\sum_{j=1}^{l} p^j(x_a^j(p) - e_a^j) \leq 0, \quad x_a(p) \geq 0.$$

Then, $x_a(p) \in M_a(p)$, so the second condition (M2) is proved. Hence, we conclude that $M_a(p_n) \to M_a(p)$ in Mosco's sense when $p_n \to p \in P$. □

In the proof of the continuity result of the solution $\overline{x}_a(p)$ on P we will use the following Lemma (e. g. see [22], Lemma 2.2), regarding the Minty variational inequality:

Lemma 2 *Let $-\nabla u_a(x_a)$ be satisfying the condition (8); then variational inequality (9) is equivalent to:*

$$\langle -\nabla u_a(x_a), x_a - \overline{x}_a(p)\rangle_l \geq 0 \qquad \forall x_a \in M_a(p). \tag{13}$$

Now we can prove the following:

Theorem 3 *Let $-\nabla u_a(x_a)$ be a continuous operators verifying conditions (8). Then the unique solution $\overline{x}_a(p) \in M_a(p)$ to variational inequality (9) is continuous on P.*

Proof Let $\overline{x}_a$ be a solution to the variational inequality (9), $\overline{x}_a : P \to \mathrm{R}_+^l$.
Let $p \in P$ fix and let $\{p_n\}_{n\in\mathrm{N}} \subset P$ be a sequence, such that $p_n \to p \in P$. For all $n \in \mathrm{N}$ we consider the variational inequality:

$$\langle -\nabla u_a(\overline{x}_a(p_n)), x_a - \overline{x}_a(p_n)\rangle_l \geq 0, \quad \forall x_a \in M_a(p_n).\text{''} \tag{14}$$

We must show that the sequence of the unique solutions of (14) converges to solution of the problem (9); namely:

$$\lim_{n\to+\infty} \overline{x}_a(p_n) = \overline{x}(p). \tag{15}$$

From the Lemma (1) the sequence $\{M_a(p_n)\}$ converges to $M_a(p)$ in the Mosco's sense; than from the condition (M_1), there exists a sequence $\{y_a(p_n)\}$ such that:

$$y_a(p_n) \in M_a(p_n)\, \forall n \in \mathrm{N}, \qquad \lim_{n\to+\infty} y_a(p_n) = \overline{x}_a(p). \tag{16}$$

Since $u_a(x_a) \in C^1(\mathrm{R}^l_+)$, we have:

$$\lim_{n\to+\infty} (-\nabla u_a(y_a(p_n))) = -\nabla u_a(\overline{x}_a(p)). \tag{17}$$

In (14), for all $n \in \mathrm{N}$, choosing $x_a(p_n) = y_a(p_n)$, it results:

$$\langle -\nabla u_a(\overline{x}_a(p_n)), y_a(p_n) - \overline{x}_a(p_n)\rangle_l \geq 0. \tag{18}$$

By the condition (B), with $x_a = x_a(p_n)$ and $y_a = y_a(p_n)$:

$$\langle -\nabla u_a(\overline{x}_a(p_n)) + \nabla u_a(y_a(p_n)), \overline{x}_a(p_n) - y_a(p_n)\rangle \geq \nu||\overline{x}_a(p_n) - y_a(p_n)||^2;$$

From the last inequality and from (18), we have:

$$\begin{aligned} &\nu||\overline{x}_a(p_n) - y_a(p_n)||^2 \leq \langle -\nabla u_a(\overline{x}_a(p_n)) + \nabla u_a(y_a(p_n)), \overline{x}_a(p_n) - y_a(p_n)\rangle \\ &= \langle -\nabla u_a(\overline{x}_a(p_n)), \overline{x}_a(p_n) - y_a(p_n)\rangle + \langle \nabla u_a(y_a(p_n)), \overline{x}_a(p_n) - y_a(p_n)\rangle \\ &\leq || - \nabla u_a(y_a(p_n))|| \cdot ||y_a(p_n) - \overline{x}_a(p_n)||, \end{aligned}$$

namely

$$||\overline{x}_a(p_n) - y_a(p_n)|| \leq \frac{|| - \nabla u_a(y_a(p_n))||}{\nu}.$$

Then, we have:

$$||\overline{x}_a(p_n)|| \leq ||\overline{x}_a(p_n) - y_a(p_n)|| + ||y_a(p_n)|| \leq \frac{|| - \nabla u_a(y_a(p_n))||}{\nu} + ||y_a(p_n)|| \tag{19}$$

From the conditions (16) and (17), there exist $h,\ k \in \mathrm{R}_+$ such that:

$$|| - \nabla u_(y_a(p_n))|| \leq h, \quad ||y_a(p_n)|| \leq k \quad \forall n \in \mathrm{N}.$$

So, from (19), it follows:

$$||\overline{x}_a(p_n)|| \leq \frac{h}{\nu} + k, \qquad \forall n \in \mathrm{N},$$

where the constant $\frac{h}{\nu} + k$ does not depend on n. Hence there exists a subsequence $\{\overline{x}_a(p_{k_n}\}$ of $\{\overline{x}_a(p_n)\}$ converging to an element $y_a \in \mathrm{R}^l_+$:

$$\lim_{n\to+\infty} \overline{x}_a(p_{k_n}) = y_a.$$

Taking into account of the condition (M_2) of the convergence in Mosco's sense, we have that $y_a \in M_a(p)$.

In virtue of the condition (M_1) of the convergence in Mosco's sense related to sets $M_a(p_n)$ it results:

$$\forall x_a \in M_a(p)\ \exists\{x_a(p_n)\} :\ x_a(p_n) \in M_a(p_n), \quad \lim_{n\to+\infty} x_a(p_n) = x_a,$$

and, since $u_a(x_a) \in C^1(\mathrm{R}^l_+)$, we have:

$$\lim_{n\to+\infty} (-\nabla u_a(x_a(p_n))) = -\nabla u_a(x_a).$$

For all $n \in \mathrm{N}$, we consider the variational inequality:

$$\langle -\nabla u_a(\overline{x}_a(p_{k_n})), x_a(p_{k_n}) - \overline{x}_a(p_{k_n})\rangle_l \geq 0, \quad \forall x_a(p_{n_k}) \in M_a(p_{k_n}),$$

passing to the limit as $n \to +\infty$:

$$\langle -\nabla u_a(x_a(p)), x_a(p) - y_a(p)\rangle_l \geq 0, \quad \forall x_a(p) \in M_a(p);$$

namely, by the Minty's Lemma, $y_a(p) \in M_a(p)$ is a solution to the variational inequality (9). By uniqueness of the solution to (9), we have $y_a(p) = \overline{x}_(p)$. Hence it follows that every subsequence of $\{\overline{x}_a(p)\}$ converges to the same limit $\overline{x}_a(p)$ and hence

$$\lim_{n\to+\infty} \overline{x}_a(p_n) = \overline{x}_a(p).$$

Then we can conclude that the solution $\overline{x}_a(p)$ to the variational inequality (9) is continuous on P.

Finally, we can prove the existence of the solution to the quasi-variational inequality (7):

Theorem 4 *Let* $(-\nabla u_a(x_a))$ *be an operator that satisfies the assumption* (8)*. Then there exists* $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ *solution to quasi variational inequality* (7)*.*

Proof Let $\overline{x}_a$ be the unique solution to the variational inequality (9). Since P is a compact and convex set and $\overline{x}_a(p)$ is a continuous function, by the Theorem 2, we have the existence of solution to the variational inequality (10). Then, the pair $(\overline{p}, \overline{x}) \in P \times M(\overline{p})$ is a solution to the quasi variational inequality (7).

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